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Fractional quantum Hall effect on the 2-sphere: a quasispin analysis

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Abstract. We study a class of models for the fractional quantum Hall effect, where the system is mapped onto the surface of a sphere around a strong magnetic monopole. When the magnetic charge of the monopole is half-integer (in units of $\hbar/e$), there exists an SU(2) quasispin group operating on the Fock space of these models. This group commutes with the group of rotations on the sphere, but not with the particle number operator N. This fact may be utilized to easily construct states of definite angular momentum at high particle numbers from states of the same angular momentum at lower particle numbers. Using the Wigner-Eckart theorem the matrix elements of the Hamiltonian between these states can also be constructed. For a large class of pair interactions this leads to a computational method of finding exact eigenstates and eigenvalues of the Hamiltonian.

1. Introduction

A convenient way to construct theoretical models for the fractional quantum Hall effect (FQHE) [1] is to replace the (say) Hall bar geometry of a heterojunction by the surface of a sphere, with the homogeneous magnetic field being created by a magnetic monopole at the centre of the sphere. This was first done by Haldane [2]. The advantage of this replacement is that it leads to the most familiar (and thus simplest) geometric symmetry group for the problem, i.e. the ordinary rotation group+. The usual Born-von Karman periodic boundary conditions (i.e. mapping the system onto a 2-torus) is of no particular advantage in this case, because the two translation operators do not commute with each other when a magnetic field is present, and because the 2-torus has a more complicated topology than the surface of a sphere.

From the point of view of numerical investigations of such FQHE models, it is of great interest to utilize the symmetries of the problem as much as possible, since this will block diagonalize the Hamiltonian, and thereby reduce the size of the matrices which must be treated numerically [3]. This leads to large savings in computer time and storage [4], which in turn increases the size of the systems which can be investigated numerically. For the FQHE the necessity of extending the computation to quite high

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+ Actually its SU(2) covering group, when the magnetic charge q of the monopole is half-integer (measured in units of $\hbar/e$).

particle numbers is immediate. As functions of the filling factor ν, no macroscopic cusps in the ground state energy per particle $\varepsilon(\nu)$ have yet been found [5]. We can estimate the size of the particle number N which is necessary to see a structure in $\varepsilon(\nu)$ on a scale $\Delta\nu$. According to Dirac's quantization condition, the charge q of a magnetic monopole (measured in units of $\hbar/e$) is either integer or half-integer. The first Landau level can host $2q+1$ particles, so the filling factor is $\nu = N/(2q+1)$. Thus, since N and $2q$ can only be varied in integer steps, a resolution in the filling factor of $\Delta\nu$ can only be achieved for particle numbers $N \geqslant \nu^2/\Delta\nu$, which may be rather large.

However, it is in general a rather non-trivial problem to explicitly construct a basis of states which are eigenstates of the total angular momentum operator, and the matrix elements of the Hamiltonian between such states. In a previous paper [6] we made a case study of how this can be done with little computational work when $N = 4$ and $J = 0$, up to fairly large values of q. The purpose of the present paper is to utilize these results to generate a set of $J = 0$ states at higher particle numbers N, and the corresponding matrix elements of the Hamiltonian between these states. For a large class of interaction potentials, exact eigenstates of the Hamiltonian can be formed from the set constructed (but these will not necessarily include the ground state). In general the method can be viewed as a convenient method for generating a basis of trial wavefunction of the appropriate symmetry class.

The method is based on the Wigner–Eckart theorem in connection with an SU(2) group which commutes with the angular momentum operators J, but not with the number operator N. This group is known in atomic theory as the quasispin group [7]. It exists whenever q is half-integer.

The rest of this paper is organized as follows. In section 2 we introduce the class of models to be studied, briefly review some basic aspects of this class, and identify the su(2) quasispin Lie algebra. In section 3 we classify the operators of the model according to their transformation properties under quasispin, and show how the standard Hamiltonians can be split into quasispin scalar, vector and tensor parts (where the vector part is essentially proportional to N). In section 4 we construct the states which diagonalize the quasispin Casimir operator $I^2 = I(I+1)$, for the cases of $I = I_{\max} = (2q+1)/4$, $I = I_{\max} - 2$, and the matrix elements of the Hamiltonian between these states. This sets the stage for the numerical analysis.

2. The two-fermion operators and the quasispin algebra

The class of models we want to study, and our basic notation, are the same as in [6], where the FQHE system was mapped onto the surface of a 2-sphere and projected onto the fully spin-polarized states of the lowest Landau level. The (rotational invariant) pair-interaction V could be expressed in terms of the two-fermion operators

$$C_{LM} = \sum_{m,n} (-)^{M-n+q}\sqrt{2L+1}\begin{pmatrix} q & q & L \\ m & -n & -M \end{pmatrix} a_m^\dagger a_n \tag{1}$$

or alternatively in terms of the set

$$\begin{aligned} A_{LM}^\dagger &= \sum_{m,n} (-)^M \sqrt{2L+1}\begin{pmatrix} q & q & L \\ m & n & -M \end{pmatrix} a_m^\dagger a_n^\dagger \\ A_{LM} &= \sum_{m,n} (-)^M \sqrt{2L+1}\begin{pmatrix} q & q & L \\ m & n & -M \end{pmatrix} a_n a_m. \end{aligned} \tag{2}$$

Here $\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$ is a Wigner $3j$-symbol, and the operators defined are spin-L spherical tensors. The A_{LM} are non-zero only when $2q+L$ is an odd integer. The creation and annihilation operators $a_n^\dagger$, a_n are related to the fermion field (projected onto the lowest Landau level) by

$$\Psi(\Omega)=\sum_{m=-q}^{q} a_m Y_{qm}^{(-q)}(\Omega) \qquad \Psi^\dagger(\Omega)=\sum_{m=-q}^{q} (-)^{m-q} a_m^\dagger Y_{q,-m}^{(q)}(\Omega) \tag{3}$$

where $Y_{jm}^{(-q)}(\Omega)$ are the monopole harmonics of Wu and Yang [8]. The pair-interaction can be expressed in the alternative forms

$$\begin{aligned} V&=\sum_L \frac{\alpha_L}{\sqrt{2L+1}} \sum_{M=-L}^{L} (-)^M :C_{LM}C_{L,-M}: \equiv \sum_L \alpha_L V_L \\ V&=-\sum_L \frac{\beta_L}{\sqrt{2L+1}} \sum_{M=-L}^{L} A_{LM}^\dagger A_{LM} \equiv \sum_L \beta_L W_L \end{aligned} \tag{4}$$

where α_L, β_L are coefficients which depend on the interaction potential between pairs of particles†. We have the relations

$$\begin{aligned} W_L&=\sum_J R_{LJ}V_J \qquad & V_L&=\sum_J R_{LJ}W_J \\ \beta_L&=\sum_J R_{LJ}\alpha_J \qquad & \alpha_L&=\sum_J R_{LJ}\beta_J \end{aligned} \tag{5}$$

where R_{LJ} is a real, symmetric and orthogonal matrix,

$$R_{LJ}=[(2L+1)(2J+1)]^{1/2}\begin{Bmatrix} q & q & L \\ q & q & J \end{Bmatrix} \tag{6}$$

with the object in curly braces a Wigner $6j$-symbol. It turns out to be useful to split R_{LJ} into an odd and an even part (depending upon whether the indices L, J are odd or even),

$$R=\begin{pmatrix} R_{++} & R_{+-} \\ R_{-+} & R_{--} \end{pmatrix}. \tag{7}$$

One may then prove the relations [6]

$$(1-R_{++})(1+2R_{++})=(1+2R_{++})(1-R_{++})=0 \tag{8}$$

from which it follows that

$$P=\tfrac{1}{3}(1+2R_{++}) \qquad \text{and} \qquad Q=\tfrac{2}{3}(1-R_{++}) \tag{9}$$

are projection operators onto the subspaces corresponding to eigenvalues 1 and $-\frac{1}{2}$ of R_{++}. We note for later use that

$$P_{L0}=\frac{1}{3}\delta_{L0}-\frac{2}{3}\frac{(2L+1)^{1/2}}{2q+1} \qquad Q_{L0}=\frac{2}{3}\delta_{L0}+\frac{2}{3}\frac{(2L+1)^{1/2}}{2q+1}. \tag{10}$$

† In addition to the pair-interaction V we shall include a particle-background and background-background interaction term $-(2q+1)^{-1}\alpha_0 N^2$ in our Hamiltonian.

The set of two-fermion operators obeys a Lie-algebra, derived in [6]

$$
\begin{aligned}
&[A_{LM}, A^\dagger_{JN}] = \theta_L\theta_J\left\{\tfrac{1}{2}\delta_{LJ}\delta_{MN} + \sum_{KR}(-)^{L+N} f_{LJK}\begin{pmatrix} L & J & K \\ -M & N & -R\end{pmatrix} C_{KR}\right\} \\
&[C_{LM}, A_{JN}] = \theta_J \sum_{KR}(-)^{L+K+N} f_{LJK}\begin{pmatrix} L & J & K \\ -M & N & -R\end{pmatrix} A_{KR} \\
&[C_{LM}, A^\dagger_{JN}] = -\theta_J \sum_{KR}(-)^{L+K+R} f_{LJK}\begin{pmatrix} L & J & K \\ M & N & -R\end{pmatrix} A^\dagger_{KR} \\
&[C_{LM}, C_{JN}] = -\sum_{KR}[1-(-)^{L+J+K}](-)^R f_{LJK}\begin{pmatrix} L & J & K \\ M & N & -R\end{pmatrix} C_{KR}
\end{aligned}
\tag{11}
$$

where $\theta_L = [1-(-)^{2q+L}]$, and $f_{LJK} = [(2L+1)(2J+1)(2K+1)]^{1/2}\{\begin{smallmatrix} L & J & K \\ q & q & q\end{smallmatrix}\}$. This is just a particular way of writing the so$(4q+2)$ algebra of bilinears in the a_n and $a_n^\dagger$—as seen from the point of view of the rotation group. There are a lot of subalgebras in the set (11), but we are only interested in those which behave simply under rotations. Apart from the angular momentum algebra itself, generated by

$$
\begin{pmatrix} C_{11} \\ C_{10} \\ C_{1,-1}\end{pmatrix} = \left[\frac{2}{q(q+1)(2q+1)}\right]^{1/2}\begin{pmatrix} -J_+/\sqrt{2} \\ J_z \\ J_-/\sqrt{2}\end{pmatrix}
$$

there is only one such obvious choice. This is the su(2) algebra generated by $A_{00} \equiv A$, $A^\dagger_{00} \equiv A^\dagger$, and $C_{00} = N/\sqrt{2q+1}$. This algebra exists whenever q is half-integer. We define the quasispin generators

$$
\begin{aligned}
&I_+ = \tfrac{1}{2}\sqrt{2q+1}\, A^\dagger = \sum_m (-)^{q-m} a^\dagger_m a^\dagger_{-m} \\
&I_- = \tfrac{1}{2}\sqrt{2q+1}\, A = \sum_m (-)^{q-m} a_{-m} a_m \\
&I_z = \tfrac{1}{2}N - \tfrac{1}{4}(2q+1) = \tfrac{1}{2}\sum_m [a^\dagger_m a_m - \tfrac{1}{2}]
\end{aligned}
\tag{12}
$$

which obey the properly normalized su(2) algebra,

$$
[I_z, I_\pm] = \pm I_\pm \qquad [I_+, I_-] = 2I_z. \tag{13}
$$

Since all these generators commute with the angular momentum operators $\boldsymbol{J}$, but not with the particle number operator N, this algebra can be used to construct states of definite angular momentum at large particle numbers from states with the same angular momentum at a lower particle number. Such states will belong to the same quasispin multiplet. Since it is in general rather difficult to directly construct states of definite angular momentum at large particle numbers, this may prove useful. In order to construct the matrix elements of the Hamiltonian between the states at higher particle numbers, the transformation properties of the Hamiltonian under quasispin is needed. This we analyse in the next section.

3. Quasispin classification of operators and Hamiltonians

In this section we classify the different operators in the model into quasispin multiplets, and show how the Hamiltonian can be split into terms with definite transformation properties under quasispin.

We start with the fermion creation and annihilation operators $a_n^\dagger$, a_n, since these are the basic building blocks for all the operators. It is straightforward to compute the following commutators

$$\begin{aligned}
&[I_z, a_m^\dagger] = \tfrac{1}{2} a_m^\dagger && [I_z, (-)^{q-m} a_{-m}] = -\tfrac{1}{2}(-)^{q-m} a_{-m} \\
&[I_+, a_m^\dagger] = 0 && [I_+, (-)^{q-m} a_{-m}] = a_m^\dagger \\
&[I^-, a_m^\dagger] = (-)^{q-m} a_{-m} && [I_-, (-)^{q-m} a_{-m}] = 0
\end{aligned} \tag{14}$$

which show that the pair

$$x_m \equiv \begin{pmatrix} x_m^{(1/2)} \\ x_m^{(-1/2)} \end{pmatrix} \equiv \begin{pmatrix} a_m^\dagger \\ (-)^{q-m} a_{-m} \end{pmatrix} \tag{15}$$

transforms as a quasispin-$\frac{1}{2}$ multiplet. Using the standard Clebsch-Gordan machinery it is now simple to construct two-fermion multiplets of definite quasispin. These will either be quasispin triplets or singlets. The triplets are constructed as

$$X_{mn}^{(\gamma)} = \sum_{\alpha\beta} \langle \tfrac{1}{2}\, \tfrac{1}{2}\, \alpha\, \beta \,|\, 1\, \gamma \rangle x_m^{(\alpha)} x_n^{(\beta)}$$

which when coupled to operators of definite angular momentum,

$$X_{LM}^{(\gamma)} = \sum_{mn} \langle q\, q\, m\, n \,|\, L\, M \rangle X_{mn}^{(\gamma)}$$

become

$$X_{LM} = \begin{pmatrix} A_{LM}^\dagger \\ -\sqrt{2}\, C_{LM} + \sqrt{q+\tfrac{1}{2}}\, \delta_{L0}\delta_{M0} \\ -(-)^M A_{L,-M} \end{pmatrix} \tag{16}$$

when L is even, and zero otherwise. The singlets are constructed as

$$Z_{mn} = \sum_{\alpha} \langle \tfrac{1}{2}\, \tfrac{1}{2}\, \alpha\, -\alpha \,|\, 0\, 0 \rangle x_m^{(\alpha)} x_n^{(-\alpha)}$$

which when coupled to operators of definite angular momentum become

$$Z_{LM} = \begin{cases} \sqrt{2}\, C_{LM} & \text{when } L \text{ is odd} \\ \sqrt{4q+2} & \text{when } L = M = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{17}$$

One may also verify directly from the commutator algebra (11) that X_{LM} and Z_{LM} have the correct transformation properties. We also note that the $L = 0$ quasispin triplet is essentially the set of quasispin generators itself,

$$\begin{pmatrix} X_{00}^{(1)} \\ X_{00}^{(0)} \\ X_{00}^{(-1)} \end{pmatrix} = -2\sqrt{\frac{2}{2q+1}} \begin{pmatrix} -I_+/\sqrt{2} \\ I_z \\ I_-/\sqrt{2} \end{pmatrix}.$$

We may now continue in the same manner to couple the multiplets (16) and (17) to angular momentum singlets of definite quasispin. These may either be singlets or tensors. With appropriate normalization the singlets can be chosen as

$$S_L = \frac{1}{\sqrt{2L+1}} \times \begin{cases} \Sigma_M \left[(-)^M C_{LM} C_{L,-M} + \tfrac{1}{2}(A_{LM}^\dagger A_{LM} + A_{LM} A_{LM}^\dagger)\right] & \text{for even } L \neq 0 \\ 4\boldsymbol{I}^2/(2q+1) & \text{for } L = 0 \\ \Sigma_M (-)^M C_{LM} C_{L,-M} & \text{for } L \text{ odd} \end{cases} \tag{18}$$

where $\boldsymbol{I}$ is the vector of quasispin generators. This can be written in normal ordered form, using

$$
\begin{aligned}
\sum_M (-)^M C_{LM} C_{L,-M}
&= \sum_M (-)^M :C_{LM} C_{L,-M}: + \frac{2L+1}{2q+1} N \\
&= \sum_M (-)^M :C_{LM} C_{L,-M}: + 2\frac{2L+1}{2q+1} I_z + \tfrac{1}{2}(2L+1)
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
\tfrac{1}{2} \sum_M (A^\dagger_{LM} A_{LM} + A_{LM} A^\dagger_{LM})
&= \sum_M A^\dagger_{LM} A_{LM} - 2\frac{2L+1}{2q+1} N + (2L+1) \\
&= \sum_M A^\dagger_{LM} A_{LM} - 4\frac{2L+1}{2q+1} I_z .
\end{aligned}
$$

This gives

$$
S_L = \begin{cases} V_L - W_L - 3Q_{L0} I_z - \tfrac{3}{4}(2q+1) P_{L0} & \text{for } L \text{ even} \\ V_L + 2\dfrac{(2L+1)^{1/2}}{2q+1} I_z + \tfrac{1}{2}(2L+1)^{1/2} & \text{for } L \text{ odd} \end{cases}
\tag{20}
$$

where P_{L0}, Q_{L0} are the matrix elements of (10). With appropriate normalizations the zeroth components of the tensor operators can be chosen as†

$$
T^0_L = \frac{1}{\sqrt{2L+1}} \sum_M [2(-)^M C_{LM} C_{L,-M} - \tfrac{1}{2}(A^\dagger_{LM} A_{LM} + A_{LM} A^\dagger_{LM})]
$$

when $L \neq 0$, and

$$
T^0_0 = -4[\boldsymbol{I}^2 - 3I_z^2]/(2q+1).
$$

These two expressions may be combined into one,

$$
T^0_L = 2V_L + W_L - 12P_{L0} I_z - \tfrac{3}{2} P_{L0}(2q+1).
\tag{21}
$$

By combining these equations we may split the different contributions to the Hamiltonian into quasispin scalar, vector and tensor parts:

$$
V_L = \begin{cases} \tfrac{1}{3}(S_L + T^0_L) + (4P_{L0} + Q_{L0}) I_z + \tfrac{3}{4}(2q+1) P_{L0} & \text{for } L \text{ even} \\ S_L - 2\dfrac{(2L+1)^{1/2}}{2q+1} I_z - \tfrac{1}{2}(2L+1)^{1/2} & \text{for } L \text{ odd} \end{cases}
\tag{22}
$$

and similarly,

$$
W_L = -\tfrac{2}{3} S_L + \tfrac{1}{3} T^0_L + (4P_{L0} - 2Q_{L0}) I_z .
\tag{23}
$$

† Recall that the tensor operators only exist when L is even.

Using the relations (5) to write $(V_L - W_L) = -\Sigma_J [1 - R_{++}]_{LJ} W_J = -\frac{3}{2}\Sigma_J Q_{LJ} W_J$, and $(2V_L + W_L) = \Sigma_J [1 + 2R_{++}]_{LJ} W_J = 3\Sigma_J P_{LJ} W_J$, where P and Q are the projection operators of (9), we may decompose W_L as follows

$$W_L = \left[\tfrac{1}{2}(2q+1)P_{L0} + 2Q_{L0}I_z + \sum_J Q_{LJ}W_J\right] + \left[-\tfrac{1}{2}(2q+1)P_{L0} - 4P_{L0}I_z + \sum_J P_{LJ}W_J\right] + [(4P_{L0} - 2Q_{L0})I_z] \tag{24}$$

where the terms enclosed in brackets are respectively the scalar, tensor and vector parts of W_L. Thus, given an interaction characterized by a set of potential coefficients β_L, we may split these as $\beta_L = \beta_L^{(S)} + \beta_L^{(T)}$, where

$$\beta_L^{(S)} = \sum_J Q_{LJ}\beta_J \qquad \beta_L^{(T)} = \sum_J P_{LJ}\beta_J \tag{25}$$

and write the pair-interaction as $V = V^{(S)} + V^{(T)} + V^{(V)}$, with

$$\begin{aligned} V^{(S)} &= \tfrac{1}{2}(2q+1)\beta_0^{(T)} + 2\beta_0^{(S)}I_z + \sum_L \beta_L^{(S)}W_L \\ V^{(T)} &= -\tfrac{1}{2}(2q+1)\beta_0^{(T)} - 4\beta_0^{(T)}I_z + \sum_L \beta_L^{(T)}W_L \\ V^{(V)} &= (4\beta_0^{(T)} - 2\beta_0^{(S)})I_z. \end{aligned} \tag{26}$$

Assume now that we have calculated the matrix elements of these operators between states of some low particle number N (i.e. with some low $I_z = I_z^0$), some definite total quasispins I, I', and some other quantum numbers α, β. By the Wigner–Eckart theorem for the matrix elements of spin-L tensor operators T_M^L,

$$\langle I\, I_z;\, \alpha|T_M^L|I'\, I_z';\, \beta\rangle = \langle I;\, \alpha\|T_M^L\|I';\, \beta\rangle\langle I\, I_z\, L\, M|I'\, I_z'\rangle$$

these matrix elements are related to the matrix elements at other particle numbers (i.e. a general quasispin I_z). The scalar part is independent of I_z, and non-zero only when $I = I'$. The tensor part varies as

$$\langle I\, I_z;\, \alpha|V^{(T)}|I'\, I_z;\, \beta\rangle = f(I, I';\, I_z, I_z^0)\langle I\, I_z^0;\, \alpha|V^{(T)}|I'\, I_z^0;\, \beta\rangle \tag{27}$$

where $f(I, I';\, I_z, I_z^0) = f(I', I;\, I_z, I_z^0) = \langle I\, I_z\, 2\, 0|I'\, I_z\rangle/\langle I\, I_z^0\, 2\, 0|I'\, I_z^0\rangle$. Explicitly:

$$\begin{aligned} f(I, I;\, I_z, I_z^0) &= \left[\frac{3I_z^2 - I(I+1)}{3I_z^{0^2} - I(I+1)}\right] \\ f(I, I+1;\, I_z, I_z^0) &= \left[\frac{(I+I_z+1)(I-I_z+1)I_z^2}{(I+I_z^0+1)(I-I_z^0+1)I_z^{0^2}}\right]^{1/2} \\ f(I, I+2;\, I_z, I_z^0) &= \left[\frac{(I+I_z+1)(I+I_z+2)(I-I_z+1)(I-I_z+2)}{(I+I_z^0+1)(I+I_z^0+2)(I-I_z^0+1)(I-I_z^0+2)}\right]^{1/2}. \end{aligned} \tag{28}$$

In the construction process we have in mind it would be natural first to construct the matrix elements of (26) at the endpoint of the lowest possible quasispin, $I_z^0 = -I$, and then use (27) to obtain the remaining matrix elements. In the large q limit the factors (28) then behave like

$$\begin{aligned} f(I, I;\, I_z, -I) &\sim (3z^2 - 1)/2 \\ f(I, I+1;\, I_z, -I) &\sim [z^2(1-z^2)]^{1/2}I \\ f(I, I+2;\, I_z, -I) &\sim [(1-z^2)/\sqrt{8}]I \end{aligned} \tag{29}$$

where $z = I_z/I$. Thus, the coupling between multiplets of different quasispin will be strongly enhanced away from the endpoints.

4. Eigenstates of total quasispin I^2

The total set of $J=0$ states can be classified into multiplets of definite quasi-spin I, as indicated schematically in figure 1. To apply the Wigner–Eckart theorem to our problem we must construct a basis of states in which I^2 is diagonal. The simplest multiplet to construct is the one to which the zero particle state $|0\rangle$ belongs. This is the multiplet which (for a given q) has the largest possible quasispin, $I=I_{\max}\equiv(2q+1)/4$. The normalized states in this multiplet are

$$|I_{\max}\, I_z\rangle=\left[\frac{(I_{\max}-I_z)!}{(2I_{\max})!(I_{\max}+I_z)!}\right]^{1/2}(I^+)^{I_{\max}+I_z}|0\rangle \tag{30}$$

describing a state with $N=2(I_{\max}+I_z)$ particles. To calculate the matrix elements of the pair interaction in this state, we first evaluate them in the zero particle state, $I_z=-I_{\max}$,

$$\begin{aligned} V_{11}^{(S)}(-I_{\max})&\equiv\langle I_{\max}\,-I_{\max}|V^{(S)}|I_{\max}\,-I_{\max}\rangle=2(\beta_0^{(T)}-\beta_0^{(S)})I_{\max}\\ V_{11}^{(T)}(-I_{\max})&\equiv\langle I_{\max}\,-I_{\max}|V^{(T)}|I_{\max}\,-I_{\max}\rangle=2\beta_0^{(T)}I_{\max}. \end{aligned} \tag{31}$$

Use of the Wigner–Eckart theorem through (27) and (28) now gives

$$\begin{aligned} V_{11}^{(S)}(I_z)&=V_{11}^{(S)}(-I_{\max})\\ V_{11}^{(T)}(I_z)&=\langle I_{\max}\,I_z|V^{(T)}|I_{\max}\,I_z\rangle=2\frac{3I_z^2-I_{\max}(I_{\max}+1)}{(2I_{\max}-1)}\beta_0^{(T)}. \end{aligned} \tag{32}$$

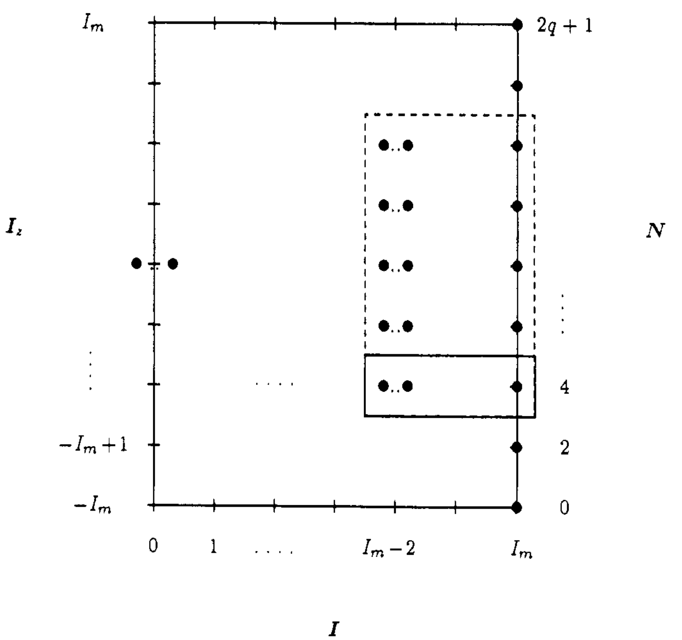

Figure 1. Schematic view of the location of states in the $I-I_z$ plane. I_m is the maximum quasispin value $(2q+1)/4$. For $N=4$ we include all the possible states, whereas for higher N, we only diagonalize within the subset of states in the broken frame. (Note that there are no $J=0$ states with quasispin $I=I_m-1$.)

Adding terms we find

$$V_{11}(I_z) \equiv \langle I_{\max}\, I_z | V | I_{\max}\, I_z \rangle$$
$$= -(2q+1)\left[\nu(\beta_0^{(S)}+\beta_0^{(T)}) - 3\nu^2\beta_0^{(T)} + \frac{6\nu(1-\nu)}{2q-1}\beta_0^{(T)}\right] \tag{33}$$

where $\nu = N/(2q+1) = (I_{\max}+I_z)/2I_{\max}$ is the filling factor. It is instructive to compare this result with the result of making a Bogoliubov transformation to a state $|\nu\rangle$ such that $\langle\nu|N|\nu\rangle = (2q+1)\nu$, i.e. $\langle I_z\rangle = (2\nu-1)I_{\max}$. This may be achieved by introducing new fermion operators

$$\begin{pmatrix} b_m^\dagger \\ (-)^{q-m}b_{-m} \end{pmatrix} = \begin{pmatrix} (1-\nu)^{1/2} & \nu^{1/2} \\ -\nu^{1/2} & (1-\nu)^{1/2} \end{pmatrix} \begin{pmatrix} a_m^\dagger \\ (-)^{q-m}a_{-m} \end{pmatrix}$$

and defining $|\nu\rangle$ to be the state which is annihilated by all the b_n. By normal ordering all expressions with respect to the b_n, $b_n^\dagger$ it is straightforward to compute matrix elements in the state $|\nu\rangle$. It is simpler, however, to make use of the fact that the Bogoliubov transformation above is simply a rotation in quasispin space around the y axis,

$$(b_n^\dagger, b_n) = U(\varphi)(a_n^\dagger, a_n)U(\varphi)^\dagger \qquad |\nu\rangle = U(\varphi)|0\rangle \tag{34}$$

where $U(\varphi) = \exp(i\varphi Iy) = \exp(-\varphi(I_+ - I_-)/2)$. This will rotate I_z in the xz plane,

$$\langle\nu|I_z|\nu\rangle = \langle 0|U(\varphi)^\dagger I_z U(\varphi)|0\rangle = \langle 0|\cos\varphi\, I_z + \sin\varphi\, I_x|0\rangle = -\cos\varphi\, I_{\max}$$

which means that we must choose $\cos\varphi = 1-2\nu$. $V^{(S)}$ is invariant under this transformation, being a scalar. Since $V^{(T)}$ is the $M=0$ member of a multiplet T_M^2 we get

$$\langle\nu|T_0^2|\nu\rangle = \langle 0|U(\varphi)^\dagger T_0^2 U(\varphi)|0\rangle = \sum_M D_{0M}^{(2)}(\varphi)\langle 0|T_M^2|0\rangle$$
$$= d_{00}^{(2)}(\varphi)\langle 0|T_0^0|0\rangle = (3\cos^2\varphi - 1)\beta_0^{(T)} I_{\max}.$$

Here we have used the fact that of the T_M^2 only T_0^2 have a non-zero expectation value in the zero particle state. Adding all contributions we obtain

$$\langle\nu|V|\nu\rangle = -(2q+1)[\nu(\beta_0^{(S)}+\beta_0^{(T)}) - 3\nu^2\beta_0^{(T)}] \tag{35}$$

which becomes identical to (33) as $q\to\infty$.

As an explicit example we take the pair potential to be the standard choice of a 3D Coulomb interaction, $V(r) = e^2/4\pi\varepsilon r$, with $r = 2l_B\sqrt{q}\sin\vartheta/2$ the chord distance (and ϑ the spherical angle) between a pair. Measuring energy in units of $e^2/4\pi\varepsilon l_B$, the corresponding potential coefficients are [6]

$$\alpha_L = \frac{A(q)}{\sqrt{2L+1}}\binom{4q+1}{2q-L} \tag{36}$$

where

$$A(q) = (2q+1)\left[2\sqrt{q}\binom{4q+1}{2q}\right]^{-1}.$$

Using properties of the matrix R_{LJ} we find

$$\beta_0^{(S)} = -\tfrac{1}{2}\sum_L{}' Q_{0L}\alpha_L + \sum_J{}' R_{0J}\alpha_J$$

$$= \frac{1}{\sqrt{q}}\left[-\frac{(2q+1)(4q+3)}{6(4q+1)} + \tfrac{1}{3}2^{4q-1}\binom{4q+1}{2q}^{-1}\right] \tag{37}$$

$$\beta_0^{(T)} = \sum_L{}' P_{0L}\alpha_L = \frac{1}{\sqrt{q}}\left[\frac{2q(2q+1)}{3(4q+1)} - \tfrac{1}{3}2^{4q-1}\binom{4q+1}{2q}^{-1}\right].$$

Here the sums over L are restricted to even L, and the sum over J is restricted to odd J. We have further used the sum formula (for $2q$ odd)

$$\sum_{n\geqslant 0}^{2q}{}' \binom{4q+1}{n} = 2^{4q-1} \mp \frac{1}{2}\frac{2q+1}{4q+1}\binom{4q+1}{2q}$$

where the sum is restricted to $n\,{}^{\text{even}}_{\text{odd}}$, and $2q$ is assumed odd. Inserting the coefficients (37) into (33), and including a particle-background and background-background interaction term $-(2q+1)^2\nu^2/\sqrt{q}$, we obtain an energy per particle

$$\varepsilon(\nu) = -\frac{1}{2\sqrt{q}}\frac{2q+1}{2q-1}\left[\frac{6q+1}{4q+1}(1-\nu) + 2^{4q}\binom{4q+1}{2q}^{-1}\left(\nu - \frac{2}{2q+1}\right)\right]. \tag{38}$$

This reproduces the previously found exact results for a completely filled Landau level ($\nu = 1$) and a two-particle state ($\nu = \nu_2 \equiv 2/(2q+1)$) [6], which is no surprise since the $I = I_{\text{max}}$-state is the only $J = 0$-state in these cases. In fact, equation (38) is simply a linear interpolation between these two cases,

$$\varepsilon(\nu) = [\varepsilon(\nu_2)(1-\nu) + \varepsilon(1)(\nu - \nu_2)]/(1-\nu_2) \tag{39}$$

and this form is valid for general interaction potentials. As $q \to \infty$, with the interaction potential (36), we obtain

$$\varepsilon(\nu) \approx -\left[\sqrt{\frac{\pi}{8}}\,\nu + \frac{3(1-\nu)}{4\sqrt{q}} + \ldots\right] \tag{40}$$

as compared with

$$\varepsilon(\nu) \approx -\left[\sqrt{\frac{\pi}{8}}\,\nu - \frac{1-\nu}{4\sqrt{q}} + \ldots\right] \tag{41}$$

while averaging the Hamiltonian in the Bogoliubov states (34). These expressions also serve as upper variational estimates, with (40) being the most accurate, as can be expected on general grounds. The q-dependence of these expressions is shown in figure 2 for $\nu = \frac{1}{3}$, together with the results obtained when including the states of the next highest quasispin. The finite-size corrections are rather large, since they decay only like $1/\sqrt{q}$. In contrast, the finite-size corrections of the semi-classical estimates in [9] decayed like $1/q$. As $q \to \infty$ the numerical values for these energies become rather poor compared with other estimates for the ground-state energy. This indicates that the weak point of the method will be to build in sufficiently good multiparticle correlations into the wavefunction. Since the state we have constructed is quite analogous to the pair-condensed state of the BCS theory of superconductivity, one suspects that a similar

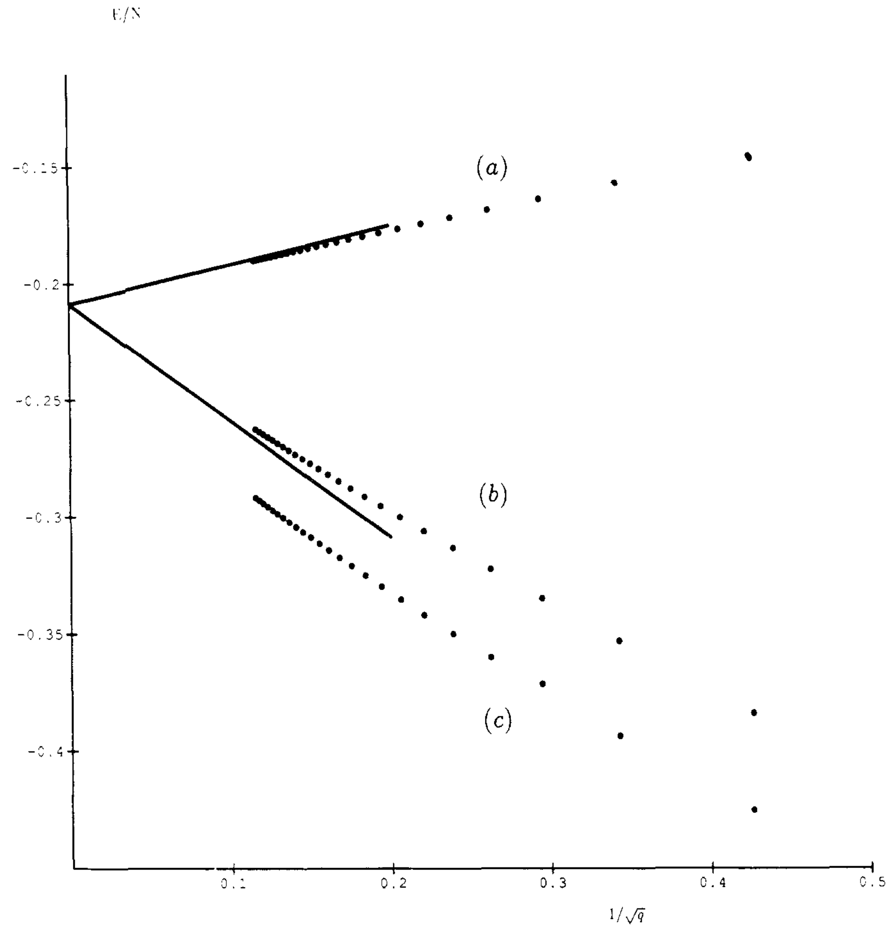

Figure 2. The finite-size behaviour of the energy per particle, $\varepsilon(\nu)$ as a function of $1/\sqrt{q}$, plotted at a filling factor $\nu \equiv N/(2q+1) = \frac{1}{3}$. The plotted energies are computed from: (a) the Bogoliubov states (34), (b) the $I_{\max}$ quasispin states (30) and (c) the states constructed from quasispin $I = (I_{\max}, I_{\max}-2)$. The asymptotic forms (41) and (40) are also indicated. This behaviour is not qualitatively different at other values of ν.

difficulty may arise in that case also (in systems where there are strong short range repulsive forces between the fermions).

We now turn to the four-particle states. In [6] we constructed an orthonormal basis of $J=0$ states as

$$|\Psi_\alpha\rangle = \sum_L \chi_L^{(\alpha)} \left[\frac{1}{\sqrt{24(2L+1)}} \sum_M (-)^M A_{LM}^\dagger A_{L,-M}^\dagger \right] |0\rangle \tag{42}$$

where (when q is half-integer) $\{\chi_L^\alpha \,|\, \alpha = 1, 2, \ldots, D \equiv [\frac{1}{3}q + \frac{1}{2}]\}$ is a complete orthonormal set of eigenvalue 1 eigenvectors for R_{++}. We also found the matrix elements of W_L between such states. For L even they are

$$\sqrt{2L+1}\langle\alpha|W_L|\beta\rangle = -12\chi_L^{(\alpha)}\chi_L^{(\beta)}. \tag{43}$$

We used a definite algorithm for generating the eigenvectors $\chi_L^{(\alpha)}$ from the matrix elements of the projection operator $P = \frac{1}{3}(1+2R_{++})$. To construct states of definite quasispin this algorithm has to be slightly modified. In equation (30) (with $I_z=-I_{\max}+2$), we have constructed a four-particle state with total quasispin $I=I_{\max}$. This corresponds to a choice

$$\chi_L^{(1)}=cP_{L0}=\tfrac{1}{3}c\left[\delta_{0L}-2\frac{(2L+1)^{1/2}}{2q+1}\right] \tag{44}$$

with $c=(1/P_{00})^{1/2}=[3(2q+1)/(2q-1)]^{1/2}$. A comparison of equations (42) and (44) with (30) does not immediately reveal that they are equivalent. However, to the vector $\chi_L^{(1)}$ we may freely add any eigenvalue $-\frac{1}{2}$ eigenvector of R_{++}, e.g. $\varphi_L=cQ_{L0}$ (because such vectors do not lead to the creation of new states, as can be seen from the calculation of matrix elements in [6]). Thus, as an alternative to equation (44) we may equivalently make the replacement

$$\chi_L^{(1)}\to\chi_L^{(1)'}=\chi_L^{(1)}+\varphi_L=c\delta_{L0}$$

in (42). With this choice it is easy to see that equation (30), with $I_z=-I_{\max}+2$, is equivalent to (42).

To proceed we construct a new projection operator

$$\bar{P}_{LJ}=\begin{cases}P_{LJ}-\dfrac{4}{3}\dfrac{[(2L+1)(2J+1)]^{1/2}}{(2q-1)(2q+1)} & \text{if } LJ\neq 0\\ 0 & \text{if } LJ=0\end{cases}$$

and decompose this as $\bar{P}_{LJ}=\Sigma_{\alpha=2}^{D}\chi_L^{(\alpha)}\chi_J^{(\alpha)}$, by the same algorithm as in [6]. The states $|\Psi_\alpha\rangle$, with $\alpha=2,\ldots,D$, become orthogonal to $|\Psi_1\rangle$, and thus are eigenstates of total quasispin with $I=I_2\equiv I_{\max}-2$. Using (43) we can construct the matrix elements

$$\begin{aligned}
V_{\alpha\beta}^{(S)}(-I_2)&\equiv\langle I_2\,-I_2;\,\alpha|V^{(S)}|I_2\,-I_2;\,\beta\rangle=V_{\alpha\beta}^{(S)}(I_z)\\
&=[2\beta_0^{(T)}I_{\max}-2\beta_0^{(S)}I_2]\delta_{\alpha\beta}-12\sum_L{}'\frac{\beta_L^{(S)}}{\sqrt{2L+1}}\chi_L^{(\alpha)}\chi_L^{(\beta)}\\
V_{\alpha\beta}^{(T)}(-I_2)&\equiv\langle I_2\,-I_2;\,\alpha|V^{(T)}|I_2\,-I_2;\,\beta\rangle\\
&=[4\beta_0^{(T)}I_2-2\beta_0^{(T)}I_{\max}]\delta_{\alpha\beta}-12\sum_L{}'\frac{\beta_L^{(T)}}{\sqrt{2L+1}}\chi_L^{(\alpha)}\chi_L^{(\beta)}\\
V_{\alpha 1}^{(T)}(-I_2)&=V_{1\alpha}^{(T)}(-I_2)\equiv\langle I_2-I_2;\,\alpha|V^{(T)}|I_{\max}-I_2;\,1\rangle\\
&=-12\sum_L{}'\frac{\beta_L^{(T)}}{\sqrt{2L+1}}\chi_L^{(\alpha)}\chi_L^{(1)}
\end{aligned} \tag{45}$$

where the sums are restricted to L even, and $\alpha,\beta=2,\ldots,D$. In addition we have already found $V_{11}^{(S,T)}(-I_2)$ in equation (32). By multiplication with the appropriate factors from (28) we can find the matrices $V_{\alpha\beta}^{(S,T)}(I_z)$ for $\alpha,\beta=1,\ldots,D$. Adding terms (and including the diagonal background term) we may easily construct the complete Hamiltonian matrix—as projected onto the states of total quasispin $I_{\max}$ and $I_{\max}-2$, for all particle numbers N. The explicit construction of this matrix, and the solution of the resulting eigenvalue problem must be done numerically. Illustrative results of this computation, for the Coulomb potential (36), are included in figure 2.

This construction will not in general lead to exact eigenstates and eigenvalues when $N>4$, because as N increases new states of lower total quasispin I are introduced, and these will couple to the states constructed here via $V^{(T)}$. However, if the interaction potential is such that the tensor part vanishes, $\beta_L^{(T)}=0$, we do get exact eigenstates and eigenvalues (but not necessarily those corresponding to the ground state for the system). Which interaction potentials will lead to a vanishing tensor part? The simplest class to describe is the one given by pair-potentials $V(\cos\vartheta)$ which satisfy the symmetry relation

$$V(\cos\vartheta)=-V(\cos(\pi-\vartheta)). \tag{46}$$

This class does not include the commonly used Coulomb interaction, which as we have seen also contains a tensor interaction part. However, in the thermodynamic limit it should not lead to a significant change of the dynamic behaviour if we modify the interaction between particles lying on opposite hemispheres in such a way that the symmetry (46) is enforced. This is so because the interaction between a given particle and an essentially homogeneous distribution of particles at a large distance will be cancelled against the interaction with the uniform background in the same region.

There is also one particular model for which the spectrum can be found exactly. This is the one described by potential coefficients $\beta_L=K\delta_{L0}$, leading to an interaction (apart from possible background terms proportional to N and N^2)

$$V=KI^+I^-=K[I^2+I_z-I_z^2]. \tag{47}$$

We note that if the constant K is positive then the state of *lowest* energy will be the one of lowest possible quasispin I, while if K is negative it will be the one of *highest* possible quasispin. Only in models which lie closer to the latter case can we expect the Wigner–Eckart method discussed in this paper to be really successful.

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